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RELATIVISTIC ELECTRON BEAM TRAJECTORIES

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ABSTRACT

A heuristic discussion is given of the influence of the self-magnetic field on the trajectory of a high current relativistic electron beam in different external magnetic fields.

This note gives a heuristic discussion of the equilibrium trajectory ($\partial/\partial t = 0$) of a high current relativistic electron beam in different external magnetic fields. For simplicity the beam is considered to be electrically neutralized by a background plasma. We show that the collective effect of the magnetic self-field of the beam may be accounted for simply by introducing an effective mass m_* equal to the sum of the electron rest mass m_o and a "magnetic mass" m_M. The approximate equation for the beam trajectory then has a form analogous to that for a single electron in a magnetic field,

$$m_* \bar{\gamma} \frac{d^2}{ds^2} \underset{\sim}{R} = \frac{q}{c^2} \left(\frac{d}{ds} \underset{\sim}{R} \right) \times \underset{\sim}{B}_{ex} \, . \tag{1}$$

As indicated in Fig. 1, $\underset{\sim}{R}(s)$ is the position vector of a point at the center of the beam and s is the arc-length measured along the beam trajectory. The effective mass $m_* = m_o + m_M$; $\underset{\sim}{B}_{ex}$ is the magnetic field owing to sources external to the beam; and $\bar{\gamma}$ is a mean Lorentz factor. We show that

$$m_M = m_o \, (I/I_A) \, \ln \, (8\mathcal{R}/a) \, , \tag{2}$$

where I is the beam current; I_A is the Alfven current; $\mathcal{R}$ is the local radius of curvature of the beam trajectory; and a is the beam radius. For $\ln (8\mathcal{R}/a) \gg 1$, as required for the validity of Eq. (2), the magnetic mass can be significant even for beam currents less than the Alfven current.

Equation (1) is obtained by considering a segment of the electron beam of length ds, as shown in Fig. 1. The average velocity of the electrons in the segment is $\langle \underset{\sim}{v} \rangle = \langle v \rangle d/ds\, \underset{\sim}{R}$, where $d/ds\, \underset{\sim}{R}$ is the unit tangent vector at the point s. For the equilibrium of the beam there is a balance of forces on the segment ds perpendicular to the beam trajectory (i.e. perpendicular to $d/ds\, \underset{\sim}{R}$). The forces parallel to the trajectory do not have an important influence for relativistic beams $\bar{\gamma}^2 \gg 1$ with $\langle v \rangle \approx c$. Three forces enter in the perpendicular force balance:

(1) The force on the segment ds (per electron) due to the external magnetic field $\underset{\sim}{B}_{ex}$:

$$\underset{\sim}{F}_{ex} = \frac{q}{c} \langle \underset{\sim}{v} \rangle \times \underset{\sim}{B}_{ex} \,. \tag{3}$$

$\underset{\sim}{B}_{ex}$ arises from currents external to the beam (including "image currents" in conducting container walls).

(2) The force on the segment ds due to the magnetic field produced by the beam current. The self-magnetic field of the beam at s is made up of a field cylindrically symmetric about the direction $d/ds\, \underset{\sim}{R}$ and a field $\underset{\sim}{B}_s$ which arises from the fact that the beam is curved. Of course, the cylindrically symmetric self-field exerts no net force on the segment ds; however, this self-field does have the essential role of "binding" the different electron orbits together to form a beam. The self-magnetic force $\underset{\sim}{F}_s$ (per electron) due to $\underset{\sim}{B}_s$ may be written as,

$$\underset{\sim}{F}_s = \frac{q}{c} \langle \underset{\sim}{v} \rangle \times \underset{\sim}{B}_s \tag{4}$$

Using Ampere's law and assuming the beam to be infinitesmally thin,

$$\underset{\sim}{B}_s = (I/c)\int d\underset{\sim}{s}' \times (\underset{\sim}{R}' - \underset{\sim}{R})\ |\underset{\sim}{R}' - \underset{\sim}{R}|^{-3} \tag{5}$$

where I is the beam current (or net current if there is partial current neutralization). This approximation of the beam as infinitesimally thin causes a logarithmic divergence of (5) at $\underset{\sim}{R} = \underset{\sim}{R}'$, which does not appear for a finite radius beam. A useful approximation[1] to $\underset{\sim}{B}_s$ is obtained by noting that the main contribution is from the range of arc-distances from $|\underset{\sim}{R}' - \underset{\sim}{R}|$ of the order of the beam radius a to $|\underset{\sim}{R}' - \underset{\sim}{R}|$ of the order of the radius of curvature $\mathcal{R} \equiv \left|d^2/ds^2\ \underset{\sim}{R}\right|^{-1}$. We assume $\mathcal{R} >> a$ and that $\mathcal{R}$ is not a rapidly varying function of $s\left(d\mathcal{R}/ds \lesssim 1\right)$. For Eq. (4) one then finds,

$$\underset{\sim}{F}_s \approx -\frac{q}{c}\langle v\rangle \frac{I}{c}\,\ln\left(\frac{8\mathcal{R}}{a}\right)\frac{d^2}{ds^2}\underset{\sim}{R}\,. \tag{6}$$

The approximation assumes the neglect of terms of order unity compared with $\ln(8\mathcal{R}/a) >> 1$. In this limit Eq. (6) is insensitive to the form of the current density profile of the beam.

(3) The centrifugal force on the segment ds due to the finite radius of curvature of the beam. The centrifugal force is in the direction of the normal vector of the curve $\underset{\sim}{R}(s)$, i.e., in the direction of $d^2/ds^2\ \underset{\sim}{R}$. Thus the centrifugal force per electron is

$$\underset{\sim}{F}_c = -\, m_o \langle \gamma v_{\parallel}^2 \rangle \frac{d^2}{ds^2}\underset{\sim}{R} \tag{7}$$

where m_o is the electron rest mass; $v_{\parallel}$ is the component of the electron velocity in the $d/ds\ \underset{\sim}{R}$ direction; and $\gamma = (1 - \underset{\sim}{v}^2/c^2)^{-\frac{1}{2}}$ is the Lorentz factor. The average indicated in Eq. (7) is over the electrons in the segment ds .

Setting the three forces to zero we obtain,

$$m_o \left[1 + (I/I_A)\ \ell n\ (8\mathcal{R}/a)\right] \frac{d^2}{ds^2} \underset{\sim}{R} = \frac{q}{c} \left(\frac{d}{ds} \underset{\sim}{R}\right) \times \underset{\sim}{B}_{ex} \qquad (8)$$

The mean Lorentz factor is defined as $\bar{\gamma} \equiv \langle\gamma v_{\parallel}^2\rangle / c\langle v\rangle$. A characteristic current (or effective Alfven current[2,3]) is defined as $I_A \equiv (mc^3/|q|)\bar{\gamma} \simeq 17{,}000$ $\bar{\gamma}$ Amp. Evidently, from Eq. (1), the effective mass is:

$$m_* = m_o + m_o(I/I_A)\ \ell n\ (8\mathcal{R}/a) \qquad (9)$$

The first term on the right corresponds to the centrifugal force (or inertial mass) and the second term to the self-magnetic force (or magnetic mass of Eq. (2)).

For an illustration of the influence of the self-magnetic force and of Eq. (1), we discuss three elementary problems which are of recent experimental interest:

(a) <u>Interaction of two beams</u>: At $z = 0$ consider two identical beams propagating parallel to the +z axis. Recent experimental studies[4] indicate that under certain conditions the beams combine after some distance of propagation. The transverse separation of the two beams is $2r(z)$ with $r(z=0) = r_o$. The force per electron on beam 1 due to beam 2 is $(q/c)\ \langle\underset{\sim}{v}\rangle \times \underset{\sim}{B}_2$, where $\underset{\sim}{B}_2$ is the magnetic field of beam 2 evaluated at the location of beam 1. The force between the nearly parallel beams is of course attractive. In order to obtain an approximate expression for $\underset{\sim}{B}_2$

we assume $dr(z)/dz$ small, in which case $|\underset{\sim}{B}_2| \approx I/cr(z)$. From Eq. (1) we obtain,

$$\frac{d^2}{dz^2} r \approx - \left(\frac{m_o}{m_*}\right) \left(\frac{I}{I_A}\right) r^{-1} \tag{10}$$

The self-magnetic force of a single beam, which enters through the factor $(m_o/m_*) < 1$, evidently inhibits the approach of the two beams. Since for $\ln (8\mathcal{R}/a) >> 1$, m_M is insensitive to $\mathcal{R}/a$, the factor $(m_o/m_*)(I/I_A)$ is treated as a constant. The solution to Eq. (10) (which involves an error integral) then indicates that the beams combine ($r(z_c) = 0$) at a propagation distance $z_c = r_o\left(\frac{\pi}{2}\right)^{1/2}\left(\frac{m_*}{m_o}\right)^{1/2}(I_A/I)^{\frac{1}{2}}$. It is assumed in the use of Eq. (9) that $\mathcal{R} < z_c$.

(b) Reflection of a beam from a conducting surface: If a high current beam is incident on a plane conducting surface at a nearly grazing angle, an image current is induced and the repulsive magnetic field of this current may reflect the beam. There is some recent experimental evidence for this type of reflection[5]. Replacing the effect of the conducting surface by an "image beam" we have a problem similar to the above beam combination problem. However, the force between the beam and its image is repulsive in that the image current flows opposite to the direction of the beam current. In place of Eq. (10) we have $d^2/dz^2\, r \approx + (m_o/m_*)(I/I_A)\ r^{-1}$, where $r(z)$ is now the distance of the beam from the conducting surface. We estimate the characteristic propagation distance required for the beam to move an appreciable distance ($\sim r_o$) away from the conducting surface as $z_c \sim r_o(m_*/m_o)^{\frac{1}{2}}(I_A/I)^{\frac{1}{2}}$, where r_o is the distance of the beam from the surface at closest approach.

(c) Helical electron beam: Consider a high current helical electron beam in an externally applied magnetic field $\underset{\sim}{B}_{ex} = B_{ex}\,\hat{\underset{\sim}{z}}$. This type of situation is thought to occur in the initial phase of the Astron experiments carried out at Cornell[6]. Consider the case with no conducting walls. From Eq. (1) the radius of the helix R is found to be

$$R = \frac{m_*\bar{\gamma}c^2}{qB_{ex}} \cos(\phi) = \frac{m_o\bar{\gamma}c^2}{qB_{ex}} \cos(\phi) \left[1 + \frac{I}{I_A} \ln\left(\frac{8\mathcal{R}}{a}\right)\right] , \tag{11}$$

where ϕ is the pitch angle of the helix, and $\mathcal{R} = R(\cos\phi)^{-2}$ is the radius of curvature of the helix. Evidently, for the same energy $\bar{\gamma}$, the self-magnetic field acts to increase the radius R . This explicit dependence of R on I/I_A has been discussed previously in detail[7] for the case of electron rings for which $\phi = 0$. We emphasize that the explicit dependence of R on I/I_A can be misleading owing to the fact that the magnetic self-field energy of the helix is built up at the expense of the electron energy $\bar{\gamma}$. This second effect has been discussed for Astron E-layers[8].

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FOOTNOTES

1. A similar technique has been applied to obtain an equation for the evolution of a curved line vortex in an incompressible fluid [H. Hasimoto, J. Fluid Mech. 51, 477 (1972) and references therein].
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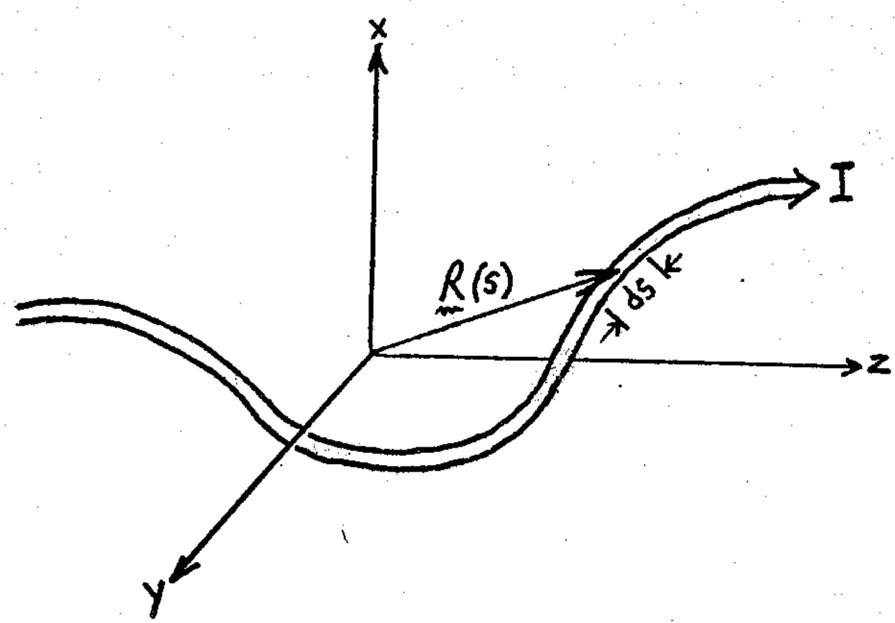

Fig. 1 - Electron beam geometry.